Semismooth Newton Method for Gradient Constrained Minimization Problem

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Problem Statement

Let $\Omega \subset \mathbb{R}^2$ be simply connected bounded Lipschitz domain. The set $K = \{v \in H_0^1(\Omega) | \quad |\nabla v| \leq 1 \text{ a.e. in } \Omega\}$ is non-empty, convex and closed in $H_0^1(\Omega)$. For a given $f \in H^{-1}(\Omega)$ we treat the variational inequality :

Problem

To find a solution $u \in K$ such that

$$\int_\Omega \nabla u \nabla (v - u) dx \geq \langle f, v \rangle \quad \forall v \in K. \tag{1}$$

This variational inequality can equivalently be formulated as a gradient constrained minimization problem:

Problem

To find a solution $u \in K$ such that

$$J(u) = \min_{v \in K} J(v)$$

where

$$J(v) = \frac{1}{2} \int_\Omega |\nabla v|^2 dx - \int_\Omega f v dx. \tag{2}$$

Regularization

- We replace constraint $|\nabla u| \leq 1$ with the equivalent constraint $|\nabla u|^2 \leq 1$
- The formal Lagrangian for the problem :

$$\mathscr{L}(v,\lambda) = \frac{1}{2}\int_{\Omega} |\nabla v|^2 dx - \int_{\Omega} fv dx + \int_{\Omega} \lambda(|\nabla v|^2 - 1) dx.$$

Now if u^* denotes a solution (the existence of which we know) then the formal Karush-Kuhn-Tucker (KKT) conditions for a Lagrange multiplier λ^* is:

$$\begin{aligned} &\int_{\Omega} (1 + 2\lambda^*)\nabla u^* \nabla v dx = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega) \\ &\lambda^* \geq 0, \quad |\nabla u^*|^2 - 1 \leq 0, \quad (\lambda^*, |\nabla u^*|^2 - 1) = 0 \end{aligned} \tag{3}$$

This system has a nonlinear structure and we want to use the Newton method for solving it. Since with the last row it is impossible to apply Newton method we reformulate the optimality system in the following way:

$$\begin{aligned} (\nabla u^*, \nabla v) + 2(\lambda^*, \nabla u^* \cdot \nabla v) = \langle f, v\rangle \quad \forall v \in H_0^1(\Omega), \\ \lambda^* = \max(0, \lambda^* + c(|\nabla u^*|^2 - 1)) \end{aligned} \tag{4}$$

where $c > 0$ is fixed and the max-operation is defined pointwise.

Newton differentiability

Definition

The mapping $F : D \subset X \to Z$ is called generalized differentiable (Newton differentiable) on the open subset $U \subset D$ if there exists a family of generalized derivatives $G : U \to L(X, Z)$ such that

$$\lim_{\|h\| \to 0} \frac{1}{\|h\|} \|F(x+h) - F(x) - G(x+h)h\| = 0,$$

for every $x \in U$.

For $\delta \in R$ we introduce the following candidate for its generalized derivative of the form:

$$G_\delta(u)(x) = \begin{cases} 1 & \text{if } u(x) > 0 \\ \delta & \text{if } u(x) = 0 \\ 0 & \text{if } u(x) < 0. \end{cases}$$

Lemma

The mapping $\max(0, \cdot) : L^q(\Omega) \to L^r(\Omega)$ *with* $1 \leq r < q \leq \infty$ *is Newton differentiable on* L^q *and* G_δ *is a generalized derivative.*

Semismooth Newton method

We use the augmented Lagrangian method for solving of the constrained minimization problem: we solve the sequence of unconstrained minimization problem with the objective functional

$$J_\gamma(v) = \frac{1}{2}\int_\Omega |\nabla v|^2 dx - \int_\Omega fv dx + \frac{\gamma}{2}\int_\Omega \max(0, (|\nabla v|^2 - 1))^2 dx.$$

to be minimized over the space $H_0^1(\Omega)$.
Further, in order to obtain the Newton differentiability we modify the problem: for $\varepsilon > 0$ sufficiently small we look for $u_{\gamma,\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega)$ minimizer of the functional

$$J_{\gamma,\varepsilon}(v) = \frac{\varepsilon}{2}(\Delta v)^2 + J_\gamma(v)$$

over the space $H^2(\Omega) \cap H_0^1(\Omega)$.

Semismooth Newton method

Problem

Find $u_{\gamma,\varepsilon} \in H^2(\Omega) \cap H_0^1(\Omega)$ *such that*

$$\varepsilon(\Delta u_{\gamma,\varepsilon}, \Delta v) + (\nabla u_{\gamma,\varepsilon}, \nabla v) + (\lambda_{\gamma,\varepsilon}, \nabla u_{\gamma,\varepsilon} \cdot \nabla v) = \langle f, v \rangle \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega),$$
$$\lambda_{\gamma,\varepsilon} = 2\gamma \max(0, |\nabla u_{\gamma,\varepsilon}|^2 - 1)).$$

For the semismooth Newton method the linearization of the nonlinear operator equation $F(u) = 0$ has the form

$$DF(u^{(k)})\delta u = -F(u^{(k)}),$$

where DF is the Newton derivative of F, δu is the update for u.

Semismooth Newton method

The use of the method leads to the following variational equation at the Newton iteration:

$$\varepsilon \int_\Omega \Delta u_{\gamma,\varepsilon}^{(k+1)} \Delta v + \int_\Omega a^{(k)} \nabla u_{\gamma,\varepsilon}^{(k+1)} \nabla v = \int_\Omega g^{(k)} \nabla u_{\gamma,\varepsilon}^{(k)} \nabla v + \int_\Omega fv, \quad \forall v \in H^2(\Omega) \cap H_0^1(\Omega) \tag{5}$$

where

$$a^{(k)} = (1 + 2\gamma\chi_{\mathscr{A}}^{(k)} \cdot (|\nabla u_{\gamma,\varepsilon}^{(k)}|^2 - 1))\mathbf{I} + 4\gamma\chi_{\mathscr{A}}^{(k)} \nabla u_{\gamma,\varepsilon}^{(k)} \otimes \nabla u_{\varepsilon}^{(k)},$$

with

$$\mathbf{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$g^{(k)} = 4\gamma\chi_{\mathscr{A}}^{(k)} |\nabla u_{\gamma,\varepsilon}^{(k)}|^2.$$

Here the characteristic function

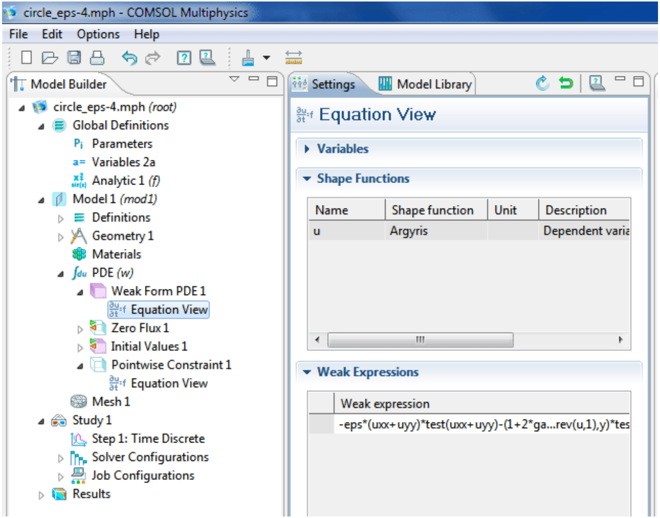
$$\chi_{\mathscr{A}}^{(k)}(x) = \begin{cases} 1 & \text{if } |\nabla u_{\gamma,\varepsilon}^{(k)}(x)| \geq 1 \\ 0 & \text{if } |\nabla u_{\gamma,\varepsilon}^{(k)}(x)| < 1 \end{cases}$$

Algorithm

Algorithm 1 Semismooth Newton Method

$\gamma := \gamma_0$, choose $u^{(0)}_{\gamma,\varepsilon}$, choose $\varepsilon > 0$
$u^{(c)} = u^{(0)}_{\gamma,\varepsilon}$
while not converged **do**
  $k = 0$
  Set $\mathscr{A}_{\gamma,0} = \left\{ x \in \Omega : |\nabla u^{(c)}|^2 > 1 \right\}$
  while not converged **do**
    $k = k + 1$
    solve (5) for $u^{(k+1)}_{\gamma,\varepsilon}$
    $\mathscr{A}_{\gamma,k+1} = \left\{ x \in \Omega : |\nabla u^{(k+1)}_{\gamma,\varepsilon}|^2 > 1 \right\}$
    if $\mathscr{A}_{\gamma,k+1} = \mathscr{A}_{\gamma,k}$ **then**
      STOP
    end if
  end while
  $u^{(c)} = u^{(k)}_{\gamma,\varepsilon}$
  increase γ
end while

Using of COMSOL Multiphysics



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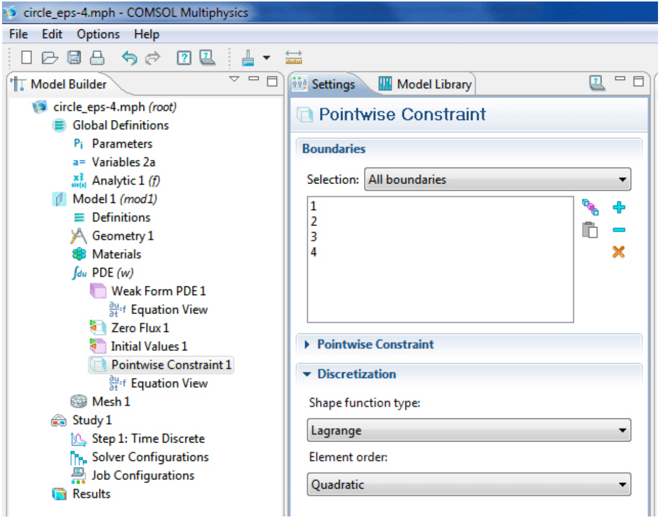

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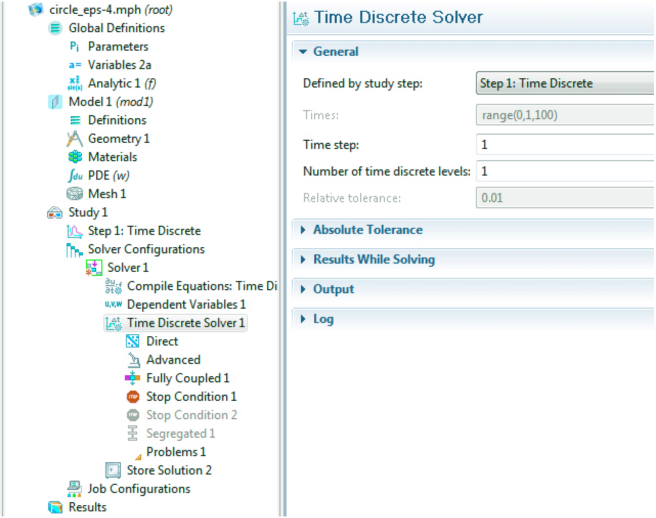

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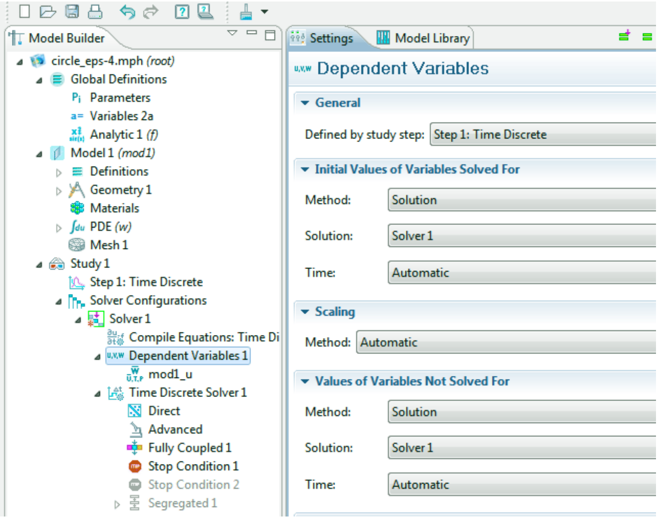

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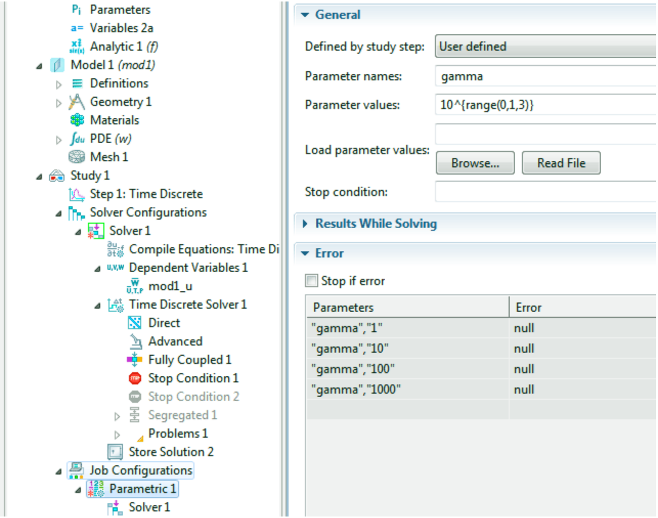


Numerical results: Example 1

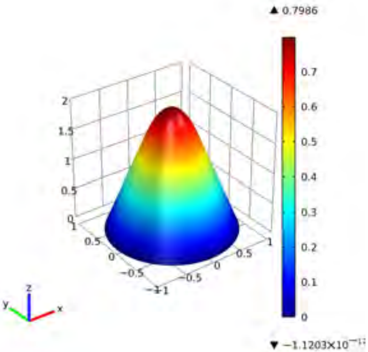
$$\min_{v\in K}\left[\frac{1}{2}\int_\Omega |\nabla v|^2 dx - d\int_\Omega v(x)dx\right]$$

$$K=\{\in H_0^1(\Omega): \quad |\nabla v|\le 1 \text{ a.e. in } \Omega\}$$

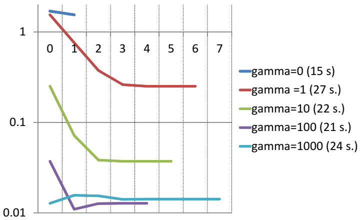
with $d=5$ and

$$\Omega=\{x\in\mathbb{R}^2| \quad x=(x_1,x_2), \quad x_1^2+x_2^2<1\}.$$

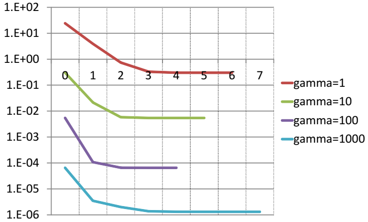
- The continuation method is initialized by zero.
- The stopping condition for the Newton iterations is $|\|u^{(k+1)}\|-\|u^{(k)}\||<10^{-10}$, where $\|u\|=(\int_\Omega|\Delta u|^2)^{\frac{1}{2}}$.



Numerical results: Example 1



(a) $|u_h - u_h^*|_{H_0^1(\Omega)}$



(b) $\int_\Omega \max(0, |\nabla u_h|^2 - 1)^2$

Figure: Convergence results for the triangulation mesh with 1902 triangles and 8873 DOF (u_h computed solution); $\varepsilon = 0.0001$; time estimation for Intel(R) Core(TM) i3 CPU 2.27 GHz

Numerical results: Example 1

To check the convergence rate of Newton iterations:

- first we increase the number of time-discrete levels up to maximal number of iterations and add the same number of nodes *Solver* $\rightarrow$ *Other* $\rightarrow$ *Store Solution*
- for each iteration except the last we compute the norm value;

$$\|u^{(k)} - u^{(M)}\| = \left(\int_{\Omega} |\Delta(u^{(k)} - u^{(M)})|^2\right)^{\frac{1}{2}},$$

where $u^{(M)}$ is the last iteration for the current value of γ. This we can do by *Results* $\rightarrow$ *Derived values* $\rightarrow$ *Surface Integration*.

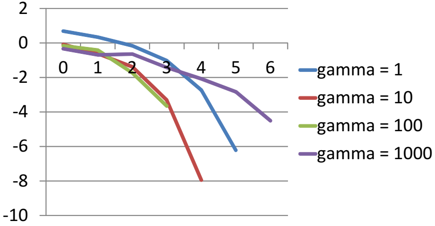


Figure: Superlinear convergence of Newton iterations (on a log-scale) with $\varepsilon = 0.0001$;

Numerical results : Example 1

To get the approximate rate of convergence we used the maximum element size in the meshes:

$$\text{convergence rate} = \log_{\frac{h_{1,\max}}{h_{2,\max}}} \frac{\text{error}_1}{\text{error}_2}$$

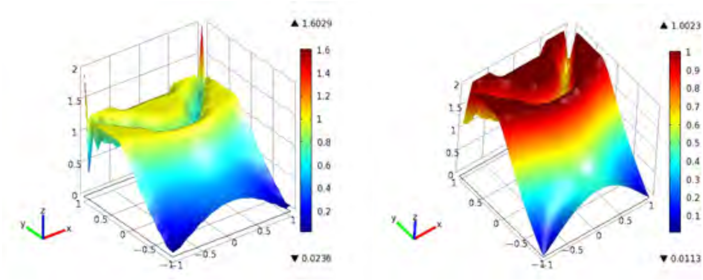
Table: Tests with various meshes; last column: convergence in $H_0^1(\Omega)$-seminorm

# of triangles	# of DOF	# of iter.	$\lvert u_h - u_h^* \rvert$	conv. rate
546	2631	13	0.0336	
936	4428	13	0.0286	0.7
1902	8873	11	0.0157	1.7
6530	29951	11	0.0067	1.4
24924	113270	12	0.0026	1.4

Numerical results: Example 2

We choose rectangular domain
$\Omega = \{x \in \mathbb{R}^2 | \quad x = (x_1, x_2), \quad -1 \le x_1 \le 1, -1 \le x_2 \le 1\}$ and

$$f(x,y) = \begin{cases} 10\cos(2((y-1)^2 + x^2 - 1)) \\ \text{if } x^2 + (y-1)^2 < 1 \\ 0 \quad \text{elsewhere} \end{cases}$$



(a) $\varepsilon = 0$ (b) $\varepsilon = 10^{-4}$

Figure: The gradient magnitude of the solution obtained on the mesh with 268 triangles and with final $\gamma = 10^3$.