Irrotational Motion of an Incompressible Fluid Past a Wing Section in an Unbounded Region

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Abstract: Developers of numerical models who address the title problem face several hurdles, such as: (1), the need to formulate boundary conditions applicable in an unbounded region; (2), The need to specify conditions suitable to ensure a unique solution in a doubly connected region; and (3), The need to allow the interior boundary to have a sharp edge, such as a cusp. The aim of the work reported herein is to build a COMSOL model tree that addresses these challenges with as few additional complications as possible and to compare the numerical results with the corresponding results of an analytical solution, specifically one associated with the names of KUTTA and ZHUKOVSKI, as described, for example, in Chapter 10 (pp 174–199) of Reference 1. Idealizations suited to this aim are: (1) Restriction to two-dimensions; (2) Exclusion of time dependencies; and (3), Exclusion of the effects of boundary-layers and wakes associated with enforcement of the no-slip boundary condition of a viscous fluid.

Keywords: Unbounded domain, Irrotational motion, Doubly connected region, KUTTA condition, Weak form, Cauchy-Riemann equations

1. Introduction

The present work employs three complex positions coordinates, namely $z = x + iy$, $Z = X + iY$, and $\zeta = \xi + i\eta$, in which (x, y), (X, Y), and (ξ, η) are the ordinary Cartesian coordinates in the complex z-, Z-, and ζ-planes, respectively. I construct a bounded neighborhood of an airfoil in the Z-plane as the image of a bounded neighborhood of a circle in the z-plane under the ZHUKOVSKI transformation

$$Z = z + a^2 / z, \tag{1.1}$$

(*e.g.* §9.4, *et seq.* of Reference 1), in which a is a constant length scale. From (1.1) we have

$$dZ/dz = 1 - a^2/z^2 . \tag{1.2}$$

Note from (1.1) and (1.2) that

$$dZ/dz = 0 \quad \text{at} \quad z = \pm a \quad \text{or} \quad Z = \pm 2a . \tag{1.3}$$

Suppose, now, that the z-plane contains a circular annulus as shown in Figure 1, which reflects a choice of center, $z_0 = x_0 + iy_0$, and inner radius of the annulus such that one of the two points where dZ/dz vanishes, namely $z = \pm a$, is situated *on*-, while the other is situated *within*, that circle.

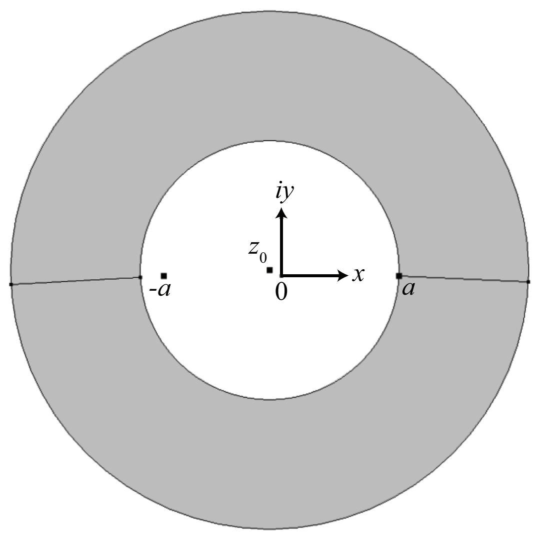

Figure 1. Annulus in the z-plane. The inner boundary intersects the point $z = a > 0$. The center of the annulus is at the point $z_0 = x_0 + iy_0$, in which $(x_0, y_0) = (-0.1a, 0.05a)$. Note that the point $z = -a < 0$ is interior to the annulus. The internal boundaries visible in the figure are radial lines that spring from z_0 and intersect the points $z = a$ and $z = -a$.

In view of (1.1) and (1.3) the TAYLOR expansion of $z \mapsto Z - 2a$ about $z = a$ must start with with

$$Z - 2a = (z - a)^2 + \ldots \tag{1.4}$$

A small circular arc of radius ε centered on the point $z = a$ and lying in the shaded region of Figure 1 will have a polar representation of the form $z - a = \varepsilon e^{i\gamma}$, the range of whose angle, γ, is about π. From (1.4) the image representation in the Z-plane is $Z - 2a \approx \varepsilon^2 e^{2i\gamma}$, the range of whose angle, 2γ, is about 2π. Thus, while the interior boundary of the annulus in Figure 1 is a curve of continuous slope its image in the Z-plane has a cusp at $Z = 2a$.

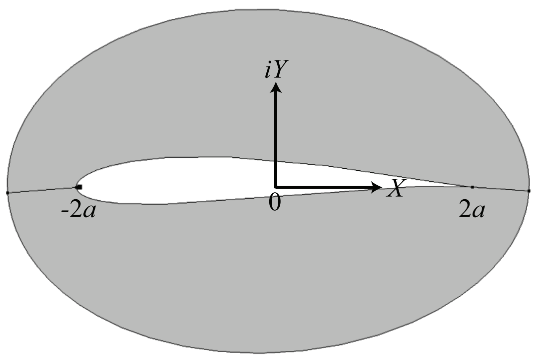

Figure 2. Image in the Z-plane of the annulus in the z-plane (Figure 1) as calculated from the ZHUKOVSKI transformation, (1.1).

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Since the exteriors of the inner circle in Figure 1 and of the airfoil in Figure 2 are both unbounded regions neither is suited for computation of the solution by the Finite Element Method. Reference 2 describes a method to address this challenge—namely by mapping an exterior region to an interior one—and the present work employs a similar artifice.

To derive a suitable exterior-to-interior mapping multiply (1.1) by Z and rearrange to get a quadratic equation for z, *viz.*

$$z^2 - Zz + a^2 = 0. \qquad (1.5)$$

If (z_1, z_2) denote the roots of (1.5) then the left member has the factorization

$$z^2 - Zz + a^2 = (z - z_1)(z - z_2). \qquad (1.6)$$

If one multiplies out the right member, cancels the term z^2 that appears on both sides, and rearranges, one obtains

$$(z_1 + z_2 - Z)z + (a^2 - z_1 z_2) = 0, \qquad (1.7)$$

which must be an identity in z. Linear independence of the distinct powers of z implies that their coefficients must vanish separately, *i.e.*

$$z_1 + z_2 = Z, \text{ and } z_1 z_2 = a^2. \qquad (1.8)_{1,2}$$

Now the exterior of the inner circle in Figure 1 must represent a set of values of one of these roots, say z_1. Then equation $(1.8)_2$ in the form $z_2 = a^2/z_1$ shows that $z_2 \to 0$ as $z_1 \to \infty$. Motivated by this observation I will call z_1 and z_2 the *exterior* and *interior* roots, respectively. Now the subscript notation for the exterior and interior roots is clumsy so I will denote them by z and ζ, respectively in what follows. According to this convention the quadratic formula yields the roots

$$z = \tfrac{1}{2}[Z + (Z^2 - 4a^2)^{1/2}], \qquad (1.9)_1$$

$$\zeta = \tfrac{1}{2}[Z - (Z^2 - 4a^2)^{1/2}], \qquad (1.9)_2$$

of the quadratic equation (1.5), or

$$z = \tfrac{1}{2}\{Z + \sqrt{R_+ R_-}\exp[i(\Theta_+ + \Theta_-)/2]\}, \qquad (1.9)_3$$

$$\zeta = \tfrac{1}{2}\{Z - \sqrt{R_+ R_-}\exp[i(\Theta_+ + \Theta_-)/2]\}, \qquad (1.9)_4$$

in which $R_\pm \exp(i\Theta_\pm)$ is the polar representation of $Z \mp 2a$. One may ensure piecewise continuous values of $\Theta_\pm$ in the upper and lower subdomains in Figure 2 by calculating them from subdomain-specific formulas, namely

$$\Theta_\pm = \arg[-i(Z \mp 2a)] + \pi/2, \qquad (1.9)_5$$

in the *upper* subdomain domain and

$$\Theta_\pm = \arg[i(Z \mp 2a)] - \pi/2 \qquad (1.9)_6$$

in the *lower* subdomain. In the mean time equation $(1.8)_2$ takes the new form

$$z\zeta = a^2, \qquad (1.10)$$

which defines the desired exterior-to-interior mapping $z \to \zeta = a^2/z$.

Consider, now, the images in the ζ-plane of the inner and outer boundaries of the annulus in the z-plane shown in Figure 1, both of which are circles. Note that if r is the radius of either of the circles in question it satisfies $|z - z_0|^2 = r^2$, or

$$(z - z_0)(z^* - z_0^*) = r^2 , \qquad (1.11)$$

in which the asterisk denotes complex conjugation. If one eliminates z from (1.11) by means of (1.10) one may arrange the result in the form

$$(\zeta - \zeta_0)(\zeta^* - \zeta_0^*) = R^2 , \qquad (1.12)_1$$

in which

$$\zeta_0 = -a^2 z_0^* / (r^2 - z_0 z_0^*) , \qquad (1.12)_2$$

and

$$R = a^2 r / (r^2 - z_0 z_0^*) . \qquad (1.12)_3$$

Now $(1.12)_1$ is equivalent to $|\zeta - \zeta_0|^2 = R^2$, which is the equation of a circle, as asserted.

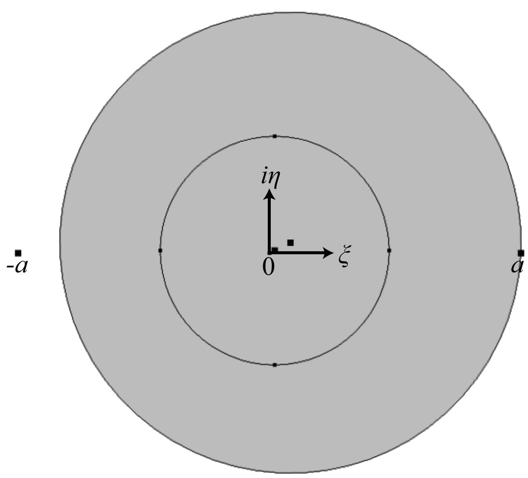

Figure 3. Image in the ζ-plane of the annulus in the z-plane (Figure 1) and its exterior, as calculated from the (1.12). The external and internal boundaries are nonconcentric circles, whose centers and radii are defined by $(1.12)_2$ and $(1.12)_3$, respectively, in which $r = b = |z_0 - a|$ and $r = b_{\mathrm{mult}} = 2b$. The internal boundary maps to the *outer* edges of the shaded regions in the Figures 1 and 2.

The whole interior of the outer circle in Figure 3 is a domain well-suited to computations since: (a), it is bounded; (b), its boundary lacks sharp edges; and (c), it is simply-connected even though its preimage in the plane of the airfoil is doubly-connected.

The z-plane, the Z-plane, and the ζ-plane define three geometries. In §2 below I will establish the systems of partial differential equations and dependent variables appropriate to them. Sections 3, 4, and 5 re-express, respectively, the impermeable-wall boundary condition, the free-stream condition, and the condition that the velocity of the fluid is finite at the trailing edge of the airfoil in the Z-plane in forms suitable for use in the ζ-plane. Section 6 presents the results.

2. Systems of Partial Differential Equations Appropriate to the Various Geometries

Let $u^{(Z)}$ and $v^{(Z)}$ denote the Cartesian components of fluid velocity in the Z-plane in the directions of the positive X- and Y-axes, respectively. I will assume that the fluid is incompressible. The local rate of expansion of the fluid must therefore vanish identically, which leads to the differential equation

$$u_X^{(Z)} + v_Y^{(Z)} = 0 . \qquad (2.1)$$

Here, and elsewhere, the subscripts denote partial differentiation (though I will make an exception to this rule in the case when subscripts distinguish components of the unit tangent vector on a boundary).

I will also assume that the motion of the fluid is *irrotational*, *i.e.* that the local *vorticity* of the fluid must vanish identically, which leads to the differential equation

$$v_X^{(Z)} - u_Y^{(Z)} = 0 . \qquad (2.2)$$

Let $(X,Y) \mapsto \psi^{(Z)}$ and $(X,Y) \mapsto \phi^{(Z)}$ denote twice differentiable functions. Then the representation

$$u^{(Z)} = \psi_Y^{(Z)} , \; v^{(Z)} = -\psi_X^{(Z)} \qquad (2.3)_{1,2}$$

satisfies (2.1) exactly, just as the representation

$$u^{(Z)} = \phi_X^{(Z)} , \; v^{(Z)} = \phi_Y^{(Z)} \qquad (2.4)_{1.2}$$

satisfies (2.2). For such solutions the dependent variables ϕ and ψ have the names *velocity potential* and *stream function*, respectively. If one eliminates $u^{(Z)}$ between $(2.3)_1$ and $(2.4)_1$ and eliminates $v^{(Z)}$ between $(2.3)_2$ and $(2.4)_2$ one obtains, respectively

$$\phi_X^{(Z)} = \psi_Y^{(Z)}\ ,\ \phi_Y^{(Z)} = -\psi_X^{(Z)} . \quad (2.5)_{1,2}$$

I assert, without proof, some basic results from the theory of functions of a complex variable, as developed, for example, in Reference 3. First, if $w = \phi + i\psi$ is the value delivered by a analytic—*i.e.* differentiable—function of a complex variable, $Z = X + iY$, and one regards the real and imaginary parts of the value (w) as functions of the real and imaginary parts of its argument (Z) then the definition of analyticity leads to the CAUCHY-RIEMANN equations, a pair of first-order equations having precisely the same structure as the system $(2.5)_{1,2}$. Since the latter represent the equations of irrotational motion of an incompressible fluid, one concludes that *any analytic function a complex variable represents a possible irrotational motion of an incompressible fluid in*, in which case the value (w) of that analytic function has the name *complex potential*.

Secondly, if two analytic functions of a complex variable are composed then the resulting composite function is also analytic: more briefly, *analyticity is preserved under composition*. In particular, the composition satisfies the CAUCHY-RIEMANN equations just as do the two functions from which it is formed.

Thirdly, *analyticity is preserved under differentiation*.

Note from $(2.4)_1$ and $(2.3)_2$, in turn, that

$$u^{(Z)} - iv^{(Z)} = \phi_X^{(Z)} + i\psi_Y^{(Z)} = w_X^{(Z)} . \quad (2.6)$$

In the mean time the derivation of the CAUCHY-RIEMANN equations employs, as an intermediate step, the equation

$$dw^{(Z)}/dZ = \partial w^{(Z)}/\partial X = w_X^{(Z)} , \quad (2.7)$$

so the last two equations imply that

$$u^{(Z)} - iv^{(Z)} = dw^{(Z)}/dZ . \quad (2.8)$$

Here, and elsewhere, I will employ the term *complex velocity* to a quantity having the structure of either the left or right member of (2.8).

If one writes down the CAUCHY-RIEMANN equations satisfied by the analytic function with value $u^{(Z)} - iv^{(Z)}$ and argument $X + iY$ one obtains a system equivalent to the system (2.1) and (2.2), which furnishes a useful check on the algebra.

In the last section I defined point-to-point formulas that allow one to map any member of the list of three complex position coordinates (z, Z, ζ) to any other. In the following I will relate the irrotational motion of an incompressible fluid in any one of the three associated complex planes to the corresponding motion in any other by the following rules. First, the values of the complex *potentials* in the z- and Z-planes, denoted by $w^{(z)}$ and $w^{(Z)}$, respectively, are equal at corresponding points, *viz*.,

$$\phi^{(z)} + i\psi^{(z)} = w^{(z)} = w^{(Z)} = \phi^{(Z)} + i\psi^{(Z)} . \quad (2.9)$$

Secondly, the values of the complex *velocities* in the z- and ζ-planes are equal at corresponding points. Thus,

$$u^{(z)} - iv^{(z)} = \frac{dw^{(z)}}{dz} = \frac{dw^{(\xi)}}{d\xi} = u^{(\xi)} - iv^{(\xi)} . \quad (2.10)$$

Since the analytic function with value $u^{(Z)} - iv^{(Z)}$ and argument $X + iY$ leads to the CAUCHY-RIEMANN equations in the form (2.1), (2.2) and analyticity is preserved under composition the analytic function with value $u^{(\zeta)} - iv^{(\zeta)}$ and argument $\xi + i\eta$ must lead to the CAUCHY-RIEMANN equations in the analogous form

$$u_\xi^{(\xi)} = -v_\eta^{(\xi)}\ ,\ u_\eta^{(\xi)} = v_\xi^{(\xi)} . \quad (2.11)_{1,2}$$

Let Ω denote the shaded region in Figure 3. Note that the functional,

$$\tfrac{1}{2}\iint_\Omega [(u_\xi^{(\xi)} + v_\eta^{(\xi)})^2 + (v_\xi^{(\xi)} - u_\eta^{(\xi)})^2]\, d\xi\, d\eta , \quad (2.12)$$

attains the minimum value zero when equations $(2.11)_{1,2}$ hold throughout Ω. If one employs the usual variational operator, δ, the first variation of (2.12) is,

$$\iint_\Omega \{(u_\xi^{(\xi)} + v_\eta^{(\xi)})[\delta(u_\xi^{(\xi)}) + \delta(v_\eta^{(\xi)})]$$

$$+(v_\xi^{(\zeta)}-u_\eta^{(\zeta)})[\delta(v_\xi^{(\zeta)})-\delta(u_\eta^{(\zeta)})]\}d\xi d\eta\,, \tag{2.13}$$

which must vanish if the expression (2.12) is to be stationary (as at a minimum). Note that the expression under the integral sign is the sum of expressions involving derivatives of $\delta(u^{(\zeta)})$, *i.e.*

$$(u_\xi^{(\zeta)}+v_\eta^{(\zeta)})\delta(u_\xi^{(\zeta)})-(v_\xi^{(\zeta)}-u_\eta^{(\zeta)})\delta(u_\eta^{(\zeta)})\,, \tag{2.14$_1$}$$

and expressions involving derivatives of $\delta(v^{(\zeta)})$, *i.e.*

$$(u_\xi^{(\zeta)}+v_\eta^{(\zeta)})\delta(v_\eta^{(\zeta)})+(v_\xi^{(\zeta)}-u_\eta^{(\zeta)})\delta(v_\xi^{(\zeta)})\,. \tag{2.14$_2$}$$

COMSOL Multiphysics has a physics interface Weak Form PDE (w). After one specifies the number of dependent variables as 2 and expands the node Weak Form PDE (w) one finds a domain-level subnode, Weak Form PDE 1. The two expressions in $(2.14)_{1,2}$ are in a form suitable for insertion in the two input fields in the Settings window for that subnode provided one writes COMSOL expressions such as `test(ux)`, `test(vx)`, `test(uy)`, *etc.* in place of the variational expressions $\delta(u_\xi^{(\zeta)}), \delta(v_\eta^{(\zeta)}), \delta(u_\eta^{(\zeta)})$, *etc.* in turn. The default Zero Flux boundary conditions are suitable here, since they amount to the condition that $(2.11)_{1,2}$ hold on the boundaries.

Given the velocity components one may apply a similar weak-form method to solve (2.3) and (2.4) for the stream function and velocity potential, respectively.

3. The Impermeable-Wall Boundary Condition

Note from $(2.3)_{1,2}$ and the general definition of the complex potential, $w = \phi + i\psi$, that

$$\det\begin{bmatrix} u^{(Z)} & v^{(Z)} \\ dX & dY \end{bmatrix} = u^{(Z)}dY - v^{(Z)}dX$$
$$= \psi_Y^{(Z)}dY + \psi_X^{(Z)}dX = d\psi^{(Z)}$$
$$= \mathrm{imag}(dw^{(Z)})\,. \tag{3.1}$$

A *streamline* is a curve that is tangent at all of its points to the local velocity vector. Now the leftmost member of (3.1) vanishes whenever a differential directed line segment with components (dX, dY) is parallel to the local velocity vector, as on a streamline. One concludes that $d\psi$ = imag(dw) vanishes along a streamline. From the middle equalities in (2.9) and (2.10) and the chain rule we have

$$dw^{(Z)} = dw^{(z)} = (dw^{(z)}/dz)dz$$
$$= (dw^{(\zeta)}/d\zeta)dz = (dw^{(\zeta)}/d\zeta)(dz/d\zeta)d\zeta\,. \tag{3.2}$$

Considering the factors in the rightmost term in turn note, first, that $dw^{(\zeta)}/d\zeta = u^{(\zeta)} - iv^{(\zeta)}$. Secondly, note from (1.10) that $z = a^2/\zeta$, so $dz/d\zeta = -a^2/\zeta^2$. Finally, note that $d\zeta = (t_\xi + it_\eta)d\sigma$, in which ($t_\xi$, t_η) are Cartesian components of the unit vector in the direction of the differential $d\zeta$ and $d\sigma$ is the corresponding arc length. Thus, the outermost equality in (3.2) becomes

$$dw^{(Z)} = (u^{(\zeta)} - iv^{(\zeta)})(-a^2/\zeta^2)(t_\xi + it_\eta)d\sigma \tag{3.3}$$

and the last equality in (3.1) becomes

$$d\psi^{(Z)} = \mathrm{imag}[(u^{(\zeta)} - iv^{(\zeta)})(1/\zeta^2)(t_\xi + it_\eta)]$$
$$\times(-a^2)d\sigma\,. \tag{3.4}$$

Taking the airfoil surface as a streamline (where $d\psi^{(Z)} = 0$) one may employ the factor in the first line of the right member of (3.4) as the constraint expression for a weak boundary constraint, and thus as a means of specifying the impermeable-wall boundary condition in the ζ-plane.

4. The Free-Stream Condition

Let $U\cos(\alpha)$ and $U\sin(\alpha)$ be the Cartesian components of the fluid velocity in the remote free stream in the Z-plane relative to the X- and Y-axes, respectively. Here U denotes the magnitude of the fluid velocity and α is the *angle of attack*. The complex velocity there is then

$$U\cos\alpha - iU\sin\alpha = Ue^{-i\alpha}\,. \tag{4.1}$$

Now the remote free stream corresponds to the limit $Z \to \infty$, or, alternatively, to the limit $z \to \infty$

in the z-plane or $\zeta \to 0$ in the ζ-plane. Let the function $\zeta \mapsto f(\zeta)$ denote the difference between the local complex velocity in the ζ-plane and the value at $\zeta = 0$, *i.e.*

$$f(\zeta) = u^{(\zeta)} - iv^{(\zeta)} - Ue^{-i\alpha}. \qquad (4.2)$$

A special case of the *Cauchy integral formula* of the theory of functions of a complex variable (See, for example, §7.4 of Reference 3) is

$$f(0) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta} d\zeta, \qquad (4.3)$$

in which the integration contour, C, is the boundary of the shaded region in Figure 3, oriented counter-clockwise. Let (t_ξ, t_η) be components of the unit tangent vector on a boundary, in this case that of the shaded region in Figure 3. Then $d\zeta = (t_\xi + it_\eta)d\sigma$ and (4.3) becomes

$$f(0) = \int_{\sigma=0}^{P} \frac{f(\zeta)(t_\xi + it_\eta)}{2\pi i\zeta} d\sigma, \qquad (4.4)$$

in which P is the circumference of C. In the Definitions list in the model for the shaded part of ζ-plane I introduced a model-coupling operator of integration type, `intop1`, which simulates the action of the integral operator in (4.4). I then employed two Global Constraints to annul the real and imaginary parts, in turn, of the right member of (4.4). Equation (4.4) affirms that the effect of these constraints is to ensure satisfaction of the free-stream condition $f(0) = 0$.

5. The Kutta condition

The double connectedness of the region surrounding the airfoil in the Z-plane permits the existence of a *family* of solutions, all of which satisfy the same impermeable-wall and free-stream boundary conditions. One may distinguish one member of this family from another by their separate values of the *circulation*, Γ, defined by

$$\Gamma = \oint_C (u^{(Z)}t_X + v^{(Z)}t_Y)ds, \qquad (5.1)$$

in which C is any closed contour that embraces the airfoil once in the *clockwise* direction, (t_X, t_Y) are components of the unit tangent vector on C, and s is an arc-length parameter.

More than a century ago W.M. KUTTA (Reference 4) found the analytic solution of a problem similar to the present one but with the airfoil replaced by a circular arc. KUTTA noticed that almost all of the solutions for arbitrary Γ exhibit infinite fluid velocity at the trailing edge. By imposing the additional condition that the velocity at the trailing edge be finite—now known as *the Kutta condition*—one can ensure a unique value for the circulation.

If one substitutes (2.9) in the form $w^{(z)} = w^{(Z)}$ into the chain rule identity $d/dz = (dZ/dz)d/dZ$ one gets $dw^{(z)}/dz = (dZ/dz)dw^{(Z)}/dZ$. But $dw^{(z)}/dz = dw^{(\zeta)}/d\zeta$ by (2.10), so

$$dw^{(\zeta)}/d\zeta = (dZ/dz)dw^{(Z)}/dZ, \quad (5.2)$$

or, in view of (1.2),

$$dw^{(\zeta)}/d\zeta = (1 - a^2/z^2)dw^{(Z)}/dZ. \quad (5.3)$$

Now the limit $Z \to 2a$ (in which $Z = 2a$ is the trailing edge of the airfoil) maps to the limits $z \to a$ and $\zeta \to a$. Meanwhile, the KUTTA condition requires that the complex velocity, $dw^{(Z)}/dZ$, in the right member of (5.3) be finite in that limit even as its coefficient in parentheses goes to zero. The right and, hence, the left member of (5.3) must, therefore, vanish in that limit, which leads to he condition

$$dw^{(\zeta)}/d\zeta = u^{(\zeta)} - iv^{(\zeta)} \to 0 \qquad (5.4)$$

as $\zeta \to a$. Condition (5.4) is suitable for use as a *weak pointwise constraint* on a boundary and suffices as an implementation of the KUTTA condition.

6. Results

Figures 4–6 illustrate computational results, via COMSOL, of the reasoning of §§1–5 above. The results in Figures 5 and 6 are graphically indistinguishable from those of the analytical solution of the same problem in Chapter 10 of Reference 1, even with COMSOL's default choice of normal mesh size. The model tree performs equally well for other choices of angle of attack, α.

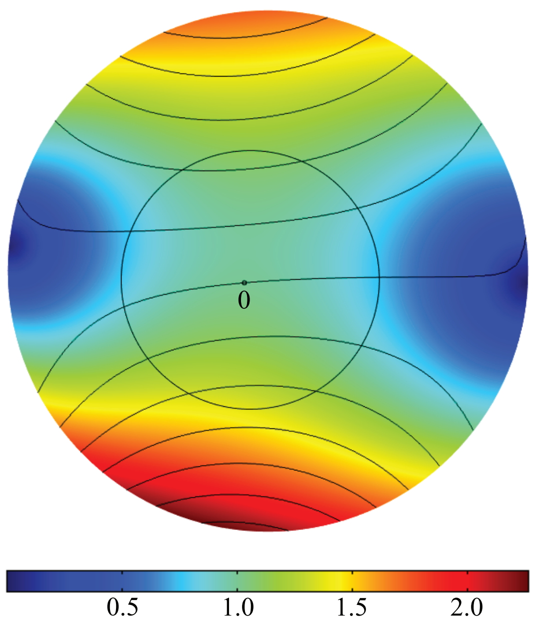

Figure 4. Distribution of $|u^{(\zeta)} - iv^{(\zeta)}|/U$ and the accompanying streamlines in the flow past a cambered ZHUKOVSKI airfoil at 5 degrees angle of attack as it appears in the ζ-plane of Figure 3. The peak value of $|u^{(\zeta)} - iv^{(\zeta)}|/U$ is 2.2643.

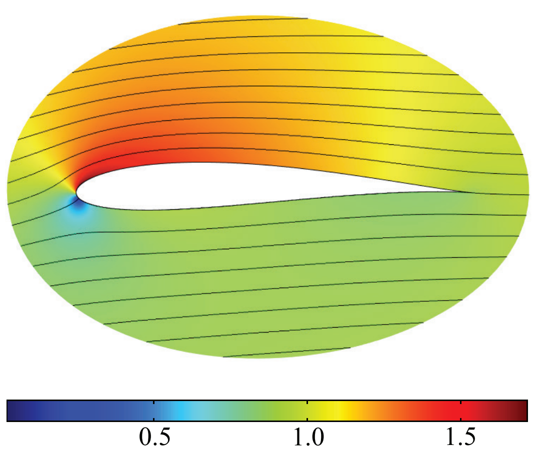

Figure 5. Distribution of $|u^{(Z)} - iv^{(Z)}|/U$, and the corresponding streamlines in the flow past a cambered ZHUKOVSKI airfoil at 5 degrees angle of attack as it appears in the Z-plane of Figure 2. The peak value of $|u^{(Z)} - iv^{(Z)}|/U$ is 1.7419.

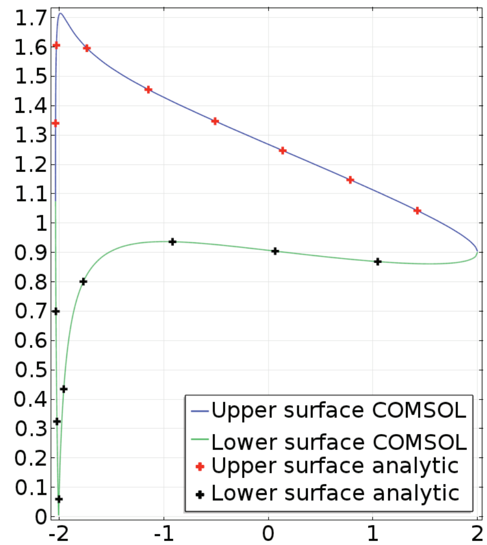

Figure 6. Distribution of $|u^{(Z)} - iv^{(Z)}|/U$ on the airfoil surface versus nondimensional chordwise position, x/a, over the upper and lower surfaces at five degrees angle of attack

7. References

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8. Acknowledgements

The author thanks Drs. ADITYA KALAVAGUNTA, and MIAN QIN, OF COMSOL for their respective versions of the two-day Intensive Course in COMSOL Multiphysics, Dr. LEI CHEN of COMSOL for her version of the Intensive Course in the CFD Module, and several members of the COMSOL online support staff for responding to his help requests.